Convection, Diffusion and Dispersion Characteristics

Physical phenomena

- Diffusion: $f_t = \alpha f_{xx}$
- Convection: $f_t + u f_x = 0$
- Dispersion: $f_t = \beta f_{xxx}$

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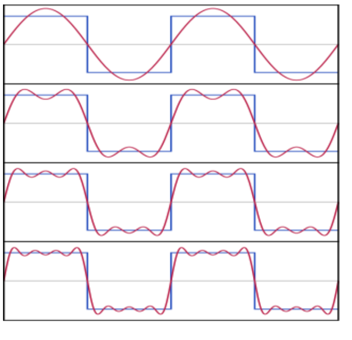
PDE = Finite Difference Solution + T.E. → Diffusion, Dispersion

Convection

$f_t + u f_x = 0$ u = Convection Velocity

Characteristics of a convection equation can be determined by investigating a simple wave moving in time and space



Convection

Using complex Fourier, a simple wave is determined as below

$$F(x,t) = Ce^{st}e^{ikx} = \Re(F(x,t)) + i\Im(F(x,t))$$

C = Wave amplitude

s = Complex wave frequency

k = Wave Number

$\Re$ = Real Part

$\Im$ = Imaginary Part

Convection

$$F(x,t) = Ce^{st}e^{ikx} = \Re(F(x,t)) + i\Im(F(x,t))$$

$$s = \sigma + i\omega$$

$$\Downarrow$$

$$F(x,t) = Ce^{\sigma t}e^{i(kx+\omega t)}$$

$$\Downarrow$$

Substituting in Convection Eq. $f_t + uf_x = 0$

$$\Downarrow$$

$$(\sigma + i\omega)Ce^{\sigma t}e^{i(kx+\omega t)} + u(ik)Ce^{\sigma t}e^{i(kx+\omega t)} = 0$$

Convection

$$(\sigma + i\omega)Ce^{\sigma t}e^{i(kx+\omega t)} + u(ik)Ce^{\sigma t}e^{i(kx+\omega t)} = 0$$

$$\Downarrow$$

$$\sigma + i(\omega + uk) = 0$$

$$\sigma = 0$$

$$\omega = -uk$$

$$\Downarrow$$

$$F(x,t) = Ce^{\sigma t}e^{i(kx+\omega t)}$$

$$\Downarrow$$

$$F(x,t) = Ce^{ik(x-ut)}$$

Convection

$$F(x,t) = Ce^{st}e^{ikx} = \Re(F(x,t)) + i\Im(F(x,t))$$

$$F(x,t) = Ce^{ik(x-ut)}$$

Consider a sine wave as Initial Condition $\quad f(x, ۰) = A_m \sin mx$

$$F(x, ۰) = Ce^{ikx} = C(\cos kx + i\sin kx) = A_m \sin mx$$

$$C = A_m$$

$$k = m$$

$$f(x,t) = A_m \sin(m(x-ut))$$

Convection

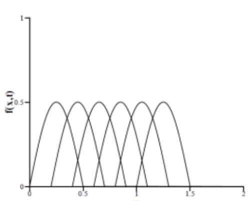
Any wave is a linear superposition of Sine and Cosine waves with different amplitudes and wave lengths

Above equations show that any Fourier component moves without any change in shape and with constant velocity

Any arbitrary wave moves without any change in shape and with constant velocity

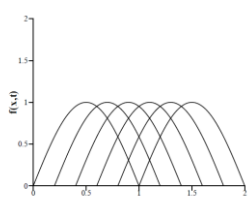
Pure Convection equation only moves the initial distribution with constant velocity

Convection



$k = ۲\pi$

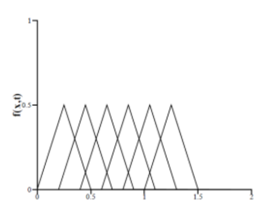
$c = u = ۰/۱$



$k = \pi$

$c = u = ۰/۱$

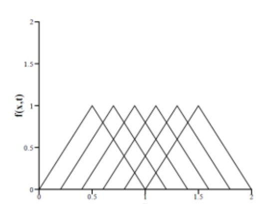
Convection



$k_۱ = ۲\pi$

$k_m = ۲m\pi$

$c = u = ۰/۱$



$k_۱ = \pi$

$k_m = m\pi$

$c = u = ۰/۱$

Diffusion

$f_t = \alpha f_{xx}$ α = **Thermal diffusion coefficient**

$$F(x,t) = Ce^{st}e^{ikx} = \Re(F(x,t)) + i\Im(F(x,t))$$
$$s = \sigma + i\omega$$

⇩

$$F(x,t) = Ce^{\sigma t}e^{i(kx+\omega t)}$$

⇩

Substituting in Convection Eq. $f_t = \alpha f_{xx}$

⇩

$$(\sigma + i\omega)Ce^{\sigma t}e^{i(kx+\omega t)} = \alpha(ik)^2 Ce^{\sigma t}e^{i(kx+\omega t)}$$

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Diffusion

$$(\sigma + i\omega)Ce^{\sigma t}e^{i(kx+\omega t)} = \alpha(ik)^2 Ce^{\sigma t}e^{i(kx+\omega t)}$$

⇩

$$\sigma + i\omega = -\alpha k^2$$
$$\sigma = -\alpha k^2$$
$$\omega = 0$$

⇩

$$F(x,t) = Ce^{-\alpha k^2 t}e^{ikx}$$

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Diffusion

$$F(x,t) = Ce^{st}e^{ikx} = \Re(F(x,t)) + i\Im(F(x,t))$$
$$F(x,t) = Ce^{-\alpha k^2 t}e^{ikx}$$

Consider a sine wave as Initial Condition $f(x,0) = A_m \sin mx$

$$F(x,0) = Ce^{ikx} = C(\cos kx + i\sin kx) = A_m \sin mx$$
$$C = A_m$$
$$k = m$$
$$f(x,t) = A_m e^{-\alpha k^2 t}\sin mx = e^{-\alpha k^2 t}(A_m \sin mx) = e^{-\alpha k^2 t}f(x,0)$$

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Diffusion

$$f(x,t) = A_m e^{-\alpha k^2 t}\sin mx = e^{-\alpha k^2 t}(A_m \sin mx) = e^{-\alpha k^2 t}f(x,0)$$

- ❖ The initial condition decreases exponentially $\exp(-\alpha k^2 t)$
- ❖ The reduction rate of initial wave depends on k^2
- ❖ The initial condition can not spread in the space

Every Fourier component decreases based on its wave number ⇨ The amplitude and shape of the initial wave changes with time

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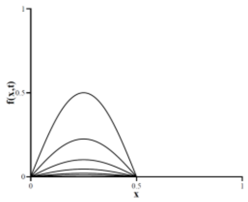
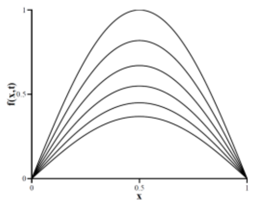
Diffusion



$k = 2\pi$

$\alpha = 0.1/\pi^2$



$k = \pi$

$\alpha = 0.1/\pi^2$

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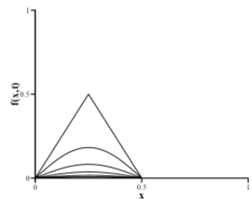
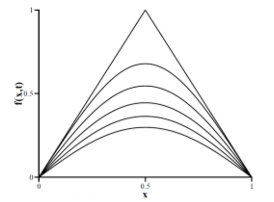
Diffusion



$k_1 = 2\pi$

$k_m = 2m\pi$

$\alpha = 0.1/\pi^2$



$k_1 = \pi$

$k_m = m\pi$

$\alpha = 0.1/\pi^2$

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Dispersion

$f_t = \beta f_{xxx}$ β = **Dispersion coefficient**

$$F(x,t) = Ce^{st}e^{ikx} = \Re(F(x,t)) + i\Im(F(x,t))$$

$$s = \sigma + i\omega$$

⇩

$$F(x,t) = Ce^{\sigma t}e^{i(kx+\omega t)}$$

⇩

Substituting in Convection Eq. $f_t = \beta f_{xxx}$

⇩

$$(\sigma + i\omega)Ce^{\sigma t}e^{i(kx+\omega t)} = \beta(ik)^3 Ce^{\sigma t}e^{i(kx+\omega t)}$$

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Dispersion

$$(\sigma + i\omega)Ce^{\sigma t}e^{i(kx+\omega t)} = \beta(ik)^3 Ce^{\sigma t}e^{i(kx+\omega t)}$$

⇩

$$\sigma + i\omega = -i\beta k^3$$

$$\sigma = 0$$

$$\omega = -\beta k^3$$

⇩

$$F(x,t) = Ce^{ik(x-\beta k^2 t)}$$

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Dispersion

$$F(x,t) = Ce^{st}e^{ikx} = \Re(F(x,t)) + i\Im(F(x,t))$$

$$F(x,t) = Ce^{ik(x-\beta k^{۲}t)}$$

Consider a sine wave as Initial Condition $\quad f(x,\circ) = A_m \sin mx$

$$F(x,\circ) = C(\cos kx + i\sin kx) = A_m \sin mx$$

$$C = A_m$$

$$k = m$$

$$f(x,t) = A_m \sin(m(x-\beta k^{۲}t))$$

Dispersion

In pure convection equation Wave velocity is constant

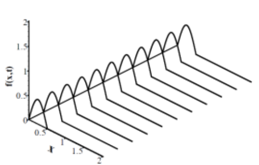
$$c = u$$

In dispersion equation Wave velocity is constant but depends on wave number

$$c = \beta k^{۲}$$

Each wave moves without any changes in its shape and amplitude but with different constant velocity ⇨ The dispersion equation spreads the initial wave in the space and changes its shape during solution time

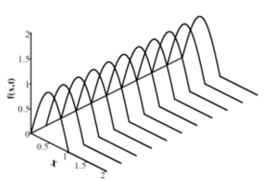
Dispersion



$k = ۲\pi$

$\beta = ۰٫۰۰۱/\pi^{۲}$

$c = \beta k^{۲} = ۰٫۰۰۴$

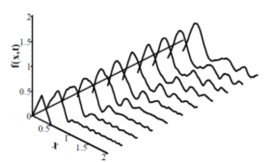


$k = \pi$

$\beta = ۰٫۰۰۱/\pi^{۲}$

$c = \beta k^{۲} = ۰٫۰۰۱$

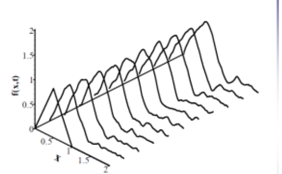
Dispersion



$k_{۱} = ۲\pi$

$k_m = ۲m\pi$

$\beta = ۰٫۰۰۱/\pi^{۲}$



$k_{۱} = \pi$

$k_m = m\pi$

$\beta = ۰٫۰۰۱/\pi^{۲}$